

ON THE ADJOINT REPRESENTATION OF \mathfrak{sl}_n AND THE FIBONACCI NUMBERS

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ABSTRACT. We decompose the adjoint representation of $\mathfrak{sl}_{r+1} = \mathfrak{sl}_{r+1}(\mathbb{C})$ by a purely combinatorial approach based on the introduction of a certain subset of the Weyl group called the *Weyl alternation set* associated to a pair of dominant integral weights. The cardinality of the Weyl alternation set associated to the highest root and zero weight of \mathfrak{sl}_{r+1} is given by the r^{th} Fibonacci number. We then obtain the exponents of \mathfrak{sl}_{r+1} from this point of view.

1. INTRODUCTION

Let G be a simple linear algebraic group over \mathbb{C} , and T a maximal algebraic torus in G of dimension r . Let $B, T \subseteq B \subseteq G$, be a choice of Borel subgroup. Then let $\mathfrak{g}, \mathfrak{h}$, and \mathfrak{b} denote the Lie algebras of G, T , and B respectively. Let Φ be the set of roots corresponding to $(\mathfrak{g}, \mathfrak{h})$, and let $\Phi^+ \subseteq \Phi$ be the choice of positive roots with respect to \mathfrak{b} . Let $P(\mathfrak{g})$ be the integral weights with respect to \mathfrak{h} , and let $P_+(\mathfrak{g})$ be the dominant integral weights. The theorem of the highest weight asserts that any finite dimensional complex irreducible representation of \mathfrak{g} is equivalent to a highest weight representation with dominant integral highest weight λ , denoted by $L(\lambda)$. A good general reference for highest weight theory in the finite dimensional setting, and for our terminology and notation is [1].

Let $W = \text{Norm}_G(T)/T$ denote the Weyl group corresponding to G and T . For $w \in W$, we let $\ell(w)$ denote the length of w . Set $\epsilon(w) = (-1)^{\ell(w)}$. Kostant's partition function is the non-negative integer valued function, \wp , defined on \mathfrak{h}^* by $\wp(\xi) =$ number of ways ξ may be written as a non-negative integral sum of positive roots, for $\xi \in \mathfrak{h}^*$.

An area of interest in combinatorial representation theory is finding the multiplicity of a weight μ in $L(\lambda)$. One way to compute this multiplicity, denoted $m(\lambda, \mu)$, is by Kostant's weight multiplicity formula [3]:

$$(1.1) \quad m(\lambda, \mu) = \sum_{\sigma \in W} \epsilon(\sigma) \wp(\sigma(\lambda + \rho) - (\mu + \rho)),$$

where $\rho = \frac{1}{2} \sum_{\alpha \in \Phi^+} \alpha$. One complication in using (1.1) to compute multiplicities is that closed formulas for the value of Kostant's partition function are not known in much generality. A second complication concerns the exponential growth of the Weyl group order

as $r \rightarrow \infty$. In practice, most terms in (1.1) are zero and hence do not contribute to the overall multiplicity. With the aim of describing the contributing terms in Kostant's weight multiplicity formula, we give the following definition.

Definition 1.1. For λ, μ dominant integral weights of \mathfrak{g} define the Weyl alternation set to be

$$\mathcal{A}(\lambda, \mu) = \{\sigma \in W \mid \wp(\sigma(\lambda + \rho) - (\mu + \rho)) > 0\}.$$

Let $\{\varpi_1, \dots, \varpi_r\}$ be the set of fundamental weights of \mathfrak{g} . Let $(,) : \mathfrak{h}^* \times \mathfrak{h}^* \rightarrow \mathbb{C}$ be the symmetric non-degenerate form corresponding to the trace form. Then the following proposition will be useful in determining Weyl alternation sets. Its proof is an easy exercise.

Proposition 1.1. Let $\xi \in \mathfrak{h}^*$. Then $\wp(\xi) > 0$ if and only if¹ $(\varpi_i, \xi) \in \mathbb{N}$, for all $1 \leq i \leq r$.

The purpose of this short note is to demonstrate that the sets $\mathcal{A}(\lambda, \mu)$ are combinatorially interesting. To this end, we specialize to the case when $\mathfrak{g} = \mathfrak{sl}_{r+1}$ and in Section 2 prove the following:

Theorem. If $r \geq 1$ and $\tilde{\alpha}$ is the highest root of \mathfrak{sl}_{r+1} , then $|\mathcal{A}(\tilde{\alpha}, 0)| = F_r$, where F_r denotes the r^{th} Fibonacci number.

This result gives rise to a (new) combinatorial identity associated to a Cartan subalgebra of \mathfrak{sl}_{r+1} , which we present in Section 3. The non-zero weights, μ , of \mathfrak{sl}_{r+1} are considered in Section 4 from the same point of view. We now introduce notation and terminology to make our approach precise.

Remark 1.1. The above theorem does not generalize to other simple Lie algebras, which motivates further research.

2. THE ZERO WEIGHT SPACE

Let $r \geq 1$, and let $n = r + 1$. Let $G = SL_n(\mathbb{C})$, $\mathfrak{g} = \mathfrak{sl}_n(\mathbb{C})$, and let²

$$\mathfrak{h} = \{\text{diag}[a_1, \dots, a_n] \mid a_1, \dots, a_n \in \mathbb{C}, \sum_{i=1}^n a_i = 0\}$$

be a fixed choice of Cartan subalgebra. Let \mathfrak{b} denote the set of $n \times n$ upper triangular complex matrices with trace zero. For $1 \leq i \leq n$, define the linear functionals $\varepsilon_i : \mathfrak{h} \rightarrow \mathbb{C}$ by $\varepsilon_i(H) = a_i$, for any $H = \text{diag}[a_1, \dots, a_n] \in \mathfrak{h}$. The Weyl group, W , is isomorphic to S_n , the symmetric group on n letters, and acts on \mathfrak{h}^* by permutations of $\varepsilon_1, \dots, \varepsilon_n$.

For each $1 \leq i \leq r$, let $\alpha_i = \varepsilon_i - \varepsilon_{i+1}$. Then the set of simple and positive roots corresponding to $(\mathfrak{g}, \mathfrak{b})$ are $\Delta = \{\alpha_1, \dots, \alpha_r\}$, and $\Phi^+ = \{\varepsilon_i - \varepsilon_j \mid 1 \leq i < j \leq n\}$ respectively. The highest root is $\tilde{\alpha} = \varepsilon_1 - \varepsilon_n = \alpha_1 + \dots + \alpha_r$. The fundamental weights are defined by $\varpi_i = \varepsilon_1 + \dots + \varepsilon_i - \frac{i}{n}(\varepsilon_1 + \dots + \varepsilon_n)$, where $1 \leq i \leq r$. Then the sets of integral weights, and dominant integral weights are

$$\begin{aligned} P(\mathfrak{g}) &= \{a_1\varpi_1 + \dots + a_r\varpi_r \mid a_i \in \mathbb{Z}, \text{ for all } i = 1, \dots, r\}, \text{ and} \\ P_+(\mathfrak{g}) &= \{a_1\varpi_1 + \dots + a_r\varpi_r \mid a_i \in \mathbb{N}, \text{ for all } i = 1, \dots, r\} \text{ respectively.} \end{aligned}$$

¹ $\mathbb{N} = \{0, 1, 2, \dots\}$.

² $\text{diag}[a_1, \dots, a_n]$ is the diagonal $n \times n$ matrix whose entries are a_1, \dots, a_n .

Let (\cdot, \cdot) be the symmetric bilinear form on \mathfrak{h}^* corresponding to the trace form as in [1]. Observe that if $H = \text{diag}[a_1, \dots, a_n] \in \mathfrak{h}$, then $(\varepsilon_1 + \dots + \varepsilon_n)(H) = a_1 + \dots + a_n = 0$. Thus for any $1 \leq i \leq r$, we can write $\varpi_i = \varepsilon_1 + \dots + \varepsilon_i$.

As a simplification we will write $\xi \in \mathfrak{h}^*$ as an n -tuple whose i -th coordinate is given by (ε_i, ξ) . Hence if $\xi = (\xi_1, \dots, \xi_n) \in \mathfrak{h}^*$ and $1 \leq i \leq r$, then $(\varpi_i, \xi) = \xi_1 + \dots + \xi_i$. Also notice $\tilde{\alpha} = (1, 0, 0, \dots, 0, -1)$ is the highest root, $\rho = (n-1, n-2, n-3, \dots, 2, 1, 0)$, and $\tilde{\alpha} + \rho = (n, n-2, n-3, \dots, 2, 1, -1)$. It will be useful to relabel $\tilde{\alpha} + \rho = (a_1, a_2, \dots, a_n)$, where

$$(2.1) \quad a_j = \begin{cases} n & \text{if } j = 1 \\ -1 & \text{if } j = n \\ n - j & \text{otherwise.} \end{cases}$$

Then for any $\sigma \in W$,

$$(2.2) \quad \sigma(\tilde{\alpha} + \rho) - \rho = (a_{\sigma^{-1}(1)} - n + 1, a_{\sigma^{-1}(2)} - n + 2, \dots, a_{\sigma^{-1}(n-1)} - 1, a_{\sigma^{-1}(n)}).$$

Theorem 2.1. *Let $\sigma \in S_n$. Then $\sigma \in \mathcal{A}(\tilde{\alpha}, 0)$ if and only if $\sigma(1) = 1$, $\sigma(n) = n$, and $|\sigma(i) - i| \leq 1$, for all $1 \leq i \leq n$.*

We begin with the following technical propositions. The proof of Proposition 2.1 is an easy exercise.

Proposition 2.1. *Let $\sigma \in S_n$. Then σ is a product of commuting neighboring transpositions if and only if $|\sigma(i) - i| \leq 1$, for all $1 \leq i \leq n$.*

Proposition 2.2. *Let $\sigma \in S_n$ such that $|\sigma(i) - i| \leq 1$, for all $1 \leq i \leq n$. Then $\sigma(1) = 1$, and $\sigma(n) = n$ if and only if for any $1 \leq i \leq r$,*

$$(\varpi_i, \sigma(\tilde{\alpha} + \rho) - \rho) = \begin{cases} 1 & \text{if } \{\sigma(1), \sigma(2), \dots, \sigma(i)\} = \{1, 2, \dots, i\} \\ 0 & \text{if } \sigma(i) = i + 1. \end{cases}$$

Proof. (\Rightarrow) Assume $\sigma \in S_n$ such that $\sigma(1) = 1$, $\sigma(n) = n$, and $|\sigma(i) - i| \leq 1$, for all $1 \leq i \leq n$. By Proposition 2.1, σ is a product of commuting neighboring transpositions, and hence $\sigma = \sigma^{-1}$. Then $\sigma(\tilde{\alpha} + \rho) - \rho = (1, 2 - \sigma(2), \dots, (n-1) - \sigma(n-1), -1)$.

We proceed by induction on i . If $i = 1$, then $\sigma(1) = 1$ and $(\varpi_1, \sigma(\tilde{\alpha} + \rho) - \rho) = 1$. Now let $1 < i \leq r$, and assume that for any $j \leq i - 1$,

$$(\varpi_j, \sigma(\tilde{\alpha} + \rho) - \rho) = \begin{cases} 1 & \text{if } \{\sigma(1), \sigma(2), \dots, \sigma(j)\} = \{1, 2, \dots, j\} \\ 0 & \text{if } \sigma(j) = j + 1. \end{cases}$$

Suppose that $j = i$. By induction hypothesis, since $j - 1 \leq i - 1$, we have that

$$(\varpi_{j-1}, \sigma(\tilde{\alpha} + \rho) - \rho) = \begin{cases} 1 & \text{if } \{\sigma(1), \sigma(2), \dots, \sigma(j-1)\} = \{1, 2, \dots, j-1\} \\ 0 & \text{if } \sigma(j-1) = j. \end{cases}$$

Case 1: Assume $(\varpi_{j-1}, \sigma(\tilde{\alpha} + \rho) - \rho) = 1$. So $\{\sigma(1), \dots, \sigma(j-1)\} = \{1, \dots, j-1\}$, and $\sigma(j) = j$ or $\sigma(j) = j + 1$. Hence,

$$(\varpi_j, \sigma(\tilde{\alpha} + \rho) - \rho) = 1 + (j - \sigma(j)) = \begin{cases} 1 & \text{if } \{\sigma(1), \dots, \sigma(j)\} = \{1, \dots, j\} \\ 0 & \text{if } \sigma(j) = j + 1. \end{cases}$$

Case 2: Assume $(\varpi_{j-1}, \sigma(\tilde{\alpha} + \rho) - \rho) = 0$. So $\sigma(j-1) = j$, and observe that

$$\begin{aligned} (\varpi_{j-1}, \sigma(\tilde{\alpha} + \rho) - \rho) &= (\varpi_{j-2}, \sigma(\tilde{\alpha} + \rho) - \rho) + ((j-1) - \sigma(j-1)) \\ &= (\varpi_{j-2}, \sigma(\tilde{\alpha} + \rho) - \rho) - 1 = 0. \end{aligned}$$

Hence $(\varpi_{j-2}, \sigma(\tilde{\alpha} + \rho) - \rho) = 1$, and by induction hypothesis we have that $\{\sigma(1), \sigma(2), \dots, \sigma(j-2)\} = \{1, 2, \dots, j-2\}$. So $\sigma(j) = j+1$ or $\sigma(j) = j-1$. If $\sigma(j) = j+1$, then $\sigma(k) = j-1$, for some integer $k \geq j+1$. This implies that $|\sigma(k) - k| \geq 2$, a contradiction. Thus $\sigma(j) = j-1$, $\{\sigma(1), \dots, \sigma(j)\} = \{1, \dots, j\}$, and $(\varpi_j, \sigma(\tilde{\alpha} + \rho) - \rho) = j - \sigma(j) = 1$.

(\Leftarrow) Let $\sigma \in S_n$ such that $|\sigma(i) - i| \leq 1$, for all $1 \leq i \leq n$. Suppose that

$$(\varpi_j, \sigma(\tilde{\alpha} + \rho) - \rho) = \begin{cases} 1 & \text{if } \{\sigma(1), \sigma(2), \dots, \sigma(j)\} = \{1, 2, \dots, j\} \\ 0 & \text{if } \sigma(j) = j+1, \end{cases} \text{ holds for any } 1 \leq j \leq r.$$

Proposition 2.1 implies that $\sigma = \sigma^{-1}$, hence (2.2) simplifies to

$$\sigma(\tilde{\alpha} + \rho) - \rho = (a_{\sigma(1)} - n + 1, a_{\sigma(2)} - n + 2, \dots, a_{\sigma(n-1)} - 1, a_{\sigma(n)}),$$

where a_i is defined by (2.1). If $\sigma(1) \neq 1$, then $\sigma(1) = 2$, and $(\varpi_1, \sigma(\tilde{\alpha} + \rho) - \rho) = -1$, a contradiction. Thus $\sigma(1) = 1$. If $\sigma(n) \neq n$, then $\sigma(n) = n-1$, and $(\varpi_r, \sigma(\tilde{\alpha} + \rho) - \rho) = -a_{\sigma(n)} = -1$, another contradiction. Thus $\sigma(n) = n$. \square

Proof of Theorem 2.1. Recall that $\sigma \in \mathcal{A}(\tilde{\alpha}, 0)$ if and only if $\wp(\sigma(\tilde{\alpha} + \rho) - \rho) > 0$. Hence it suffices to show that $\wp(\sigma(\tilde{\alpha} + \rho) - \rho) > 0$ if and only if $\sigma(1) = 1$, $\sigma(n) = n$, and $|\sigma(i) - i| \leq 1$, for all $1 \leq i \leq n$.

(\Rightarrow) Let $\sigma \in S_n$ such that $\wp(\sigma(\tilde{\alpha} + \rho) - \rho) > 0$. So, by Proposition 1.1, $(\varpi_i, \sigma(\tilde{\alpha} + \rho) - \rho) \in \mathbb{N}$, for all $1 \leq i \leq r$. By (2.2) we have that

$$\sigma(\tilde{\alpha} + \rho) - \rho = (a_{\sigma^{-1}(1)} - n + 1, a_{\sigma^{-1}(2)} - n + 2, \dots, a_{\sigma^{-1}(n-1)} - 1, a_{\sigma^{-1}(n)}),$$

where a_i is defined by (2.1). We want to prove $\sigma(1) = 1$, $\sigma(n) = n$, and $|i - \sigma(i)| \leq 1$, for all $1 \leq i \leq n$.

If $1 < \sigma^{-1}(1) \leq n$, then $(\varpi_1, \sigma(\tilde{\alpha} + \rho) - \rho) = a_{\sigma^{-1}(1)} - n + 1 < 0$, a contradiction. So $\sigma^{-1}(1) = 1$, and hence $\sigma(1) = 1$. If $1 < \sigma^{-1}(n) < n$, then $(\varpi_r, \sigma(\tilde{\alpha} + \rho) - \rho) = -a_{\sigma^{-1}(n)} = \sigma^{-1}(n) - n < 0$, a contradiction. So $\sigma^{-1}(n) = n$, and hence $\sigma(n) = n$.

Hence (2.2) simplifies to

$$\sigma(\tilde{\alpha} + \rho) - \rho = (1, 2 - \sigma^{-1}(2), 3 - \sigma^{-1}(3), \dots, (n-1) - \sigma^{-1}(n-1), -1).$$

Observe that if $|i - \sigma^{-1}(i)| \leq 1$, for all $1 < i < n$, then Proposition 2.1 implies $\sigma^{-1} = \sigma$ and thus $|i - \sigma(i)| \leq 1$, for all $1 < i < n$. Thus it suffices to show that $|i - \sigma^{-1}(i)| \leq 1$, for all $1 < i < n$.

We proceed by induction on i . If $i = 2$, then $(\varpi_2, \sigma(\tilde{\alpha} + \rho) - \rho) = 1 + 2 - \sigma^{-1}(2) \geq 0$ if and only if $\sigma^{-1}(2) \leq 3$. Since $\sigma^{-1}(1) = 1$, we have that $\sigma^{-1}(2) = 2$ or $\sigma^{-1}(2) = 3$ and in either case $|2 - \sigma^{-1}(2)| \leq 1$.

Now let $2 \leq i \leq n-1$ and assume that $|j - \sigma^{-1}(j)| \leq 1$ holds for any $j < i$.

Suppose that $j = i$. Since $j-1 < i$, we have that $|(j-1) - \sigma^{-1}(j-1)| \leq 1$. Thus $\sigma^{-1}(j-1) = j-2$, $j-1$ or j . If $\sigma^{-1}(j-1) = j-2$ or $j-1$, then $\{\sigma^{-1}(1), \dots, \sigma^{-1}(j-1)\} = \{1, \dots, j-1\}$ and $(\varpi_{j-1}, \sigma(\tilde{\alpha} + \rho) - \rho) = 1$. Hence $(\varpi_j, \sigma(\tilde{\alpha} + \rho) - \rho) = 1 + j - \sigma^{-1}(j) \geq 0$ if and only if $\sigma^{-1}(j) \leq j+1$. Thus $\sigma^{-1}(j) = j$ or $j+1$, and in either case $|j - \sigma^{-1}(j)| \leq 1$.

Now suppose that $\sigma^{-1}(j-1) = j$. Since $j-2 < i$, we have that $|(j-2) - \sigma^{-1}(j-2)| \leq 1$. Hence $\{\sigma^{-1}(1), \dots, \sigma^{-1}(j-2)\} = \{1, \dots, j-2\}$ and $(\varpi_{j-1}, \sigma(\tilde{\alpha} + \rho) - \rho) = 0$. Then

$(\varpi_j, \sigma(\tilde{\alpha} + \rho) - \rho) = j - \sigma^{-1}(j) \geq 0$ if and only if $\sigma^{-1}(j) \leq j$. Thus $\sigma^{-1}(j) = j - 1$ and $|j - \sigma^{-1}(j)| \leq 1$, which completes our induction step.

(\Leftarrow) Let $\sigma \in S_n$ such that $\sigma(1) = 1$, $\sigma(n) = n$, and $|\sigma(i) - i| \leq 1$, for all $1 \leq i \leq n$. Proposition 2.2 implies that $(\varpi_i, \sigma(\tilde{\alpha} + \rho) - \rho) = 0$ or 1 , for all $1 \leq i \leq r$. Therefore, by Proposition 1.1, $\wp(\sigma(\tilde{\alpha} + \rho) - \rho) > 0$. \square

Definition 2.1. *The Fibonacci numbers are the sequence of numbers, $\{F_n\}_{n=1}^\infty$, defined by the recurrence relation*

$$F_n = F_{n-1} + F_{n-2} \text{ for } n \geq 3, \text{ and } F_1 = F_2 = 1.$$

Remark 2.1. *The Fibonacci numbers are well known for their prevalence throughout mathematics. We refer the reader to [6].*

We leave the proof of the following lemma to the reader.

Lemma 2.1. *If $m \geq 1$, then $|\{\sigma \in S_m : |\sigma(i) - i| \leq 1, \text{ for all } 1 \leq i \leq m\}| = F_{m+1}$.*

Now for the main result of this section.

Theorem 2.2. *If $r \geq 1$ and $\tilde{\alpha}$ is the highest root of \mathfrak{sl}_{r+1} , then $|\mathcal{A}(\tilde{\alpha}, 0)| = F_r$.*

Proof. By Theorem 2.1 we know that

$$\mathcal{A}(\tilde{\alpha}, 0) = \{\sigma \in S_{r+1} \mid \sigma(1) = 1, \sigma(r+1) = r+1, \text{ and } |\sigma(i) - i| \leq 1, \forall 1 \leq i \leq r+1\}.$$

Notice that the sets $\mathcal{A}(\tilde{\alpha}, 0)$ and $\{\sigma \in S_{r-1} : |\sigma(i) - i| \leq 1, \text{ for all } 1 \leq i \leq r-1\}$ have the same cardinality. Therefore, by Lemma 2.1, $|\mathcal{A}(\tilde{\alpha}, 0)| = F_r$. \square

Remark 2.2. *For $1 \leq i \leq r$, let s_i denote the simple root reflection corresponding to $\alpha_i \in \Delta$. Then $s_i(\varepsilon_k) = \varepsilon_{\sigma(k)}$, where σ is the neighboring transposition $(i \ i+1) \in S_n$. Notice $s_i s_j = s_j s_i$ if and only if i and j are non-consecutive integers between 1 and r . Then the following corollary describes the elements of $\mathcal{A}(\tilde{\alpha}, 0)$ as products of commuting simple root reflections.*

Corollary 2.1. *Let $\sigma \in S_n$. Then $\sigma \in \mathcal{A}(\tilde{\alpha}, 0)$ if and only if $\sigma = s_{i_1} s_{i_2} \cdots s_{i_k}$, for some non-consecutive integers $2 \leq i_1, \dots, i_k \leq r-1$.*

Proof. The corollary follows from Theorem 2.1, and Proposition 2.1. \square

3. A q -ANALOG

The q -analog of Kostant's partition function is the polynomial valued function, \wp_q , defined on \mathfrak{h}^* by

$$\wp_q(\xi) = c_0 + c_1 q + \cdots + c_k q^k,$$

where c_j = number of ways to write ξ as a non-negative integral sum of exactly j positive roots, for $\xi \in \mathfrak{h}^*$. Lusztig introduced the q -analog of Kostant's weight multiplicity formula by defining a polynomial, which when evaluated at 1 gives the multiplicity of the dominant weight μ in the irreducible module $L(\lambda)$ [5]. This formula is given by

$$m_q(\lambda, \mu) = \sum_{\sigma \in W} \epsilon(\sigma) \wp_q(\sigma(\lambda + \rho) - (\mu + \rho)).$$

Let \mathfrak{g} be a simple Lie algebra of rank r . In the case when $\tilde{\alpha}$ is the highest root of \mathfrak{g} , it is known that $m_q(\tilde{\alpha}, 0) = \sum_{i=1}^r q^{e_i}$, where e_1, \dots, e_r are the exponents of \mathfrak{g} [4]. For $r \geq 1$, the exponents of \mathfrak{sl}_{r+1} are $1, 2, \dots, r$ [2]. In this section we use the Weyl alternation set $\mathcal{A}(\tilde{\alpha}, 0)$ to give a combinatorial proof of:

Theorem 3.1. *If $\tilde{\alpha}$ is the highest root of \mathfrak{sl}_{r+1} , then $m_q(\tilde{\alpha}, 0) = q + q^2 + \dots + q^r$.*

Corollary 3.1. *If $\tilde{\alpha}$ is the highest root of \mathfrak{sl}_{r+1} , then $m(\tilde{\alpha}, 0) = r$.*

Proof. This follows from Theorem 3.1 and the fact that $m_q(\tilde{\alpha}, 0)|_{q=1} = m(\tilde{\alpha}, 0)$. \square

Throughout the remainder of this section let $r \geq 1$, and let $\tilde{\alpha}$ denote the highest root of \mathfrak{sl}_{r+1} . We leave the proofs of Lemmas 3.1 and 3.2 to the reader.

Lemma 3.1. *Let $\sigma = s_1 s_2 \dots s_k \in \mathcal{A}(\tilde{\alpha}, 0)$, where i_1, \dots, i_k are non-consecutive integers between 2 and $r - 1$. Then $\sigma(\tilde{\alpha} + \rho) - \rho = \tilde{\alpha} - \sum_{j=1}^k \alpha_{i_j}$.*

Lemma 3.2. *The cardinality of the set $\{\sigma \in \mathcal{A}(\tilde{\alpha}, 0) \mid \ell(\sigma) = k\}$ is $\binom{r-1-k}{k}$, and $\max\{\ell(\sigma) \mid \sigma \in \mathcal{A}(\tilde{\alpha}, 0)\} = \lfloor \frac{r-1}{2} \rfloor$.*

We now prove the following combinatorial identity:

Proposition 3.1. *If $\sigma \in \mathcal{A}(\tilde{\alpha}, 0)$, then $\wp_q(\sigma(\tilde{\alpha} + \rho) - \rho) = q^{1+\ell(\sigma)}(1+q)^{r-1-2\ell(\sigma)}$.*

Proof. If $\sigma \in \mathcal{A}(\tilde{\alpha}, 0)$ with $\ell(\sigma) = 0$, then $\sigma = 1$ and $\sigma(\tilde{\alpha} + \rho) - \rho = \tilde{\alpha} = \alpha_1 + \dots + \alpha_r$. Since $\Phi^+ = \{\alpha_i : 1 \leq i \leq r\} \cup \{\alpha_i + \dots + \alpha_j : 1 \leq i < j \leq r\}$, for any $i \geq 0$, we can think of c_{i+1} , the coefficient of q^{i+1} in $\wp_q(\alpha_1 + \dots + \alpha_r)$, as the number of ways to place i lines in $r - 1$ slots. Hence $c_{i+1} = \binom{r-1}{i}$ and $\wp_q(\tilde{\alpha}) = \sum_{i=0}^{r-1} \binom{r-1}{i} q^{i+1} = q(1+q)^{r-1}$.

If $\sigma \in \mathcal{A}(\tilde{\alpha}, 0)$ with $\ell(\sigma) = k \neq 0$, then Corollary 2.1 implies that $\sigma = s_1 s_2 \dots s_k$, for some non-consecutive integers $2 \leq i_1, i_2, \dots, i_k \leq r - 1$. Then by Lemma 3.1, $\sigma(\tilde{\alpha} + \rho) - \rho = \tilde{\alpha} - \sum_{j=1}^k \alpha_{i_j}$. Let c_j denote the coefficient of q^j in $\wp_q(\sigma(\tilde{\alpha} + \rho) - \rho)$. Since σ subtracts k many non-consecutive simple roots from $\tilde{\alpha}$, we will at a minimum need $k + 1$ positive roots to write $\tilde{\alpha} - \sum_{j=1}^k \alpha_{i_j}$. So $c_j = 0$, whenever $j < k + 1$. Also observe that $\tilde{\alpha} - \sum_{j=1}^k \alpha_{i_j}$ can be written with at most $r - k$ positive roots. Hence $c_j = 0$, whenever $j > r - k$.

For $i \geq 0$, we can think of c_{k+1+i} as the number of ways to place i lines in $r - 1 - 2k$ slots. This is because for each simple root that σ removes from $\tilde{\alpha}$, we lose 2 slots in which to place a line, one before and one after. So $c_{k+1+i} = \binom{r-1-2k}{i}$, whenever $0 \leq i \leq r - 1 - 2k$.

Therefore $\wp_q(\sigma(\tilde{\alpha} + \rho) - \rho) = \sum_{i=0}^{r-1-2k} \binom{r-1-2k}{i} q^{k+1+i} = q^{1+k}(1+q)^{r-1-2k}$. \square

We obtain the following closed formula of Kostant's partition function by setting $q = 1$ in Proposition 3.1.

Lemma 3.3. *If $\sigma \in \mathcal{A}(\tilde{\alpha}, 0)$, then $\wp(\sigma(\tilde{\alpha} + \rho) - \rho) = 2^{r-1-2\ell(\sigma)}$.*

The following proposition will be used in the proof of Theorem 3.1.

Proposition 3.2. *For $r \geq 1$, $\sum_{k=0}^{\lfloor \frac{r-1}{2} \rfloor} (-1)^k \binom{r-1-k}{k} q^{1+k} (1+q)^{r-1-2k} = \sum_{i=1}^r q^i$.*

Proof. Equation (4.3.7) in [7] shows that for integers k and $n \geq 0$,

$$(3.1) \quad \sum_{k \leq \frac{n}{2}} (-1)^k \binom{n-k}{k} q^k (1+q)^{n-2k} = \frac{1-q^{n+1}}{1-q}.$$

Suppose $r \geq 1$, and let $n = r - 1 \geq 0$. Then by (3.1) we have that

$$\sum_{k=0}^{\lfloor \frac{r-1}{2} \rfloor} (-1)^k \binom{r-1-k}{k} q^{1+k} (1+q)^{r-1-2k} = q \left(\frac{1-q^{n+1}}{1-q} \right).$$

Now observe that $\sum_{i=1}^r q^i = \sum_{i=1}^{n+1} q^i = q \sum_{i=0}^n q^i = q \left(\frac{1-q^{n+1}}{1-q} \right).$

Therefore $\sum_{k=0}^{\lfloor \frac{r-1}{2} \rfloor} (-1)^k \binom{r-1-k}{k} q^{1+k} (1+q)^{r-1-2k} = \sum_{i=1}^r q^i.$ □

Remark 3.1. Suppose $r \geq 1$. If we define $F_r(t) = \sum_{k=0}^{\infty} \binom{r-1-k}{k} t^k$, then $F_r(1)$ is the r^{th} Fibonacci number. So $F_r(t)$ is a t -analog of the Fibonacci numbers. Also notice if $t = \frac{-q}{(1+q)^2}$, then $q(1+q)^{r-1} F_r(t)$ is the sum we encountered in Proposition 3.2.

Proof of Theorem 3.1. By Lemma 3.2 and Propositions 3.1 and 3.2, if $k = \ell(\sigma)$, then

$$\begin{aligned} m_q(\tilde{\alpha}, 0) &= \sum_{\sigma \in W} \epsilon(\sigma) \wp_q(\sigma(\tilde{\alpha} + \rho) - \rho) \\ &= \sum_{\sigma \in \mathcal{A}(\tilde{\alpha}, 0)} \epsilon(\sigma) \wp_q(\sigma(\tilde{\alpha} + \rho) - \rho) \\ &= \sum_{k=0}^{\lfloor \frac{r-1}{2} \rfloor} (-1)^k \binom{r-1-k}{k} q^{1+k} (1+q)^{r-1-2k} \\ &= q + q^2 + q^3 + \cdots + q^r. \end{aligned}$$

□

4. NON-ZERO WEIGHT SPACES

It is fundamental in Lie theory that the zero weight space is a Cartan subalgebra. If $\tilde{\alpha}$ is the highest root of \mathfrak{g} , then the non-zero weights of $L(\tilde{\alpha})$, the adjoint representation of \mathfrak{g} , are the roots and have multiplicity 1. We visit this picture from our point of view in the case when $\mathfrak{g} = \mathfrak{sl}_{r+1}$. Let $r \geq 1$, and $n = r + 1$.

Theorem 4.1. If $\mu \in P_+(\mathfrak{sl}_n)$ and $\mu \neq 0$, then $\mathcal{A}(\tilde{\alpha}, \mu) = \begin{cases} \{1\} & \text{if } \mu = \tilde{\alpha} \\ \emptyset & \text{otherwise.} \end{cases}$

We begin by proving the following propositions.

Proposition 4.1. If $\tilde{\alpha}$ is the highest root of \mathfrak{sl}_n , then $\mathcal{A}(\tilde{\alpha}, \tilde{\alpha}) = \{1\}$.

Proof. It suffices to show that $\wp(\sigma(\tilde{\alpha} + \rho) - \rho - \tilde{\alpha}) > 0$ if and only if $\sigma = 1$.

(\Rightarrow) Assume that $\sigma \in S_n$ such that $\wp(\sigma(\tilde{\alpha} + \rho) - \rho - \tilde{\alpha}) > 0$. Proposition 1.1 implies that $(\varpi_i, \sigma(\tilde{\alpha} + \rho) - \rho - \tilde{\alpha}) \in \mathbb{N}$, for all $1 \leq i \leq r$. By (2.2) we have that

$$\sigma(\tilde{\alpha} + \rho) - \rho - \tilde{\alpha} = (a_{\sigma^{-1}(1)} - n, a_{\sigma^{-1}(2)} - n + 2, \dots, a_{\sigma^{-1}(n-1)} - 1, a_{\sigma^{-1}(n)} + 1),$$

where a_i is given by (2.1).

Let $M = \{i \mid \sigma^{-1}(i) \neq i\}$. Suppose $M \neq \emptyset$, and let $j = \min(M)$. Hence $\sigma^{-1}(j) = k$, for some integer $j < k \leq n$. Since $\sigma^{-1}(i) = i$, for all $1 \leq i \leq j-1$, and by definition of a_i , we have that $(\varpi_i, \sigma(\tilde{\alpha} + \rho) - \rho - \tilde{\alpha}) = 0$, for all $1 \leq i \leq j-1$.

Thus,

$$(\varpi_j, \sigma(\tilde{\alpha} + \rho) - \rho - \tilde{\alpha}) = a_{\sigma^{-1}(j)} - n + j = \begin{cases} j - n - 1 & \text{if } k = n \\ j - k & \text{if } j < k < n. \end{cases}$$

In either case $(\varpi_j, \sigma(\tilde{\alpha} + \rho) - \rho - \tilde{\alpha}) < 0$, a contradiction. Thus $M = \emptyset$, and $\sigma^{-1}(i) = i$, for all $1 \leq i \leq n$. Therefore $\sigma = 1$.

(\Leftarrow) If $\sigma = 1$, then $\wp(\sigma(\tilde{\alpha} + \rho) - \rho - \tilde{\alpha}) = \wp(0) = 1 > 0$. \square

Proposition 4.2. *Let $\mu \in P_+(\mathfrak{sl}_n)$, and $\mu \neq 0$. Then there exists $\sigma \in S_n$ such that $\wp(\sigma(\tilde{\alpha} + \rho) - \rho - \mu) > 0$ if and only if $\mu = \tilde{\alpha}$.*

Proof. (\Rightarrow) If $\mu \in P_+(\mathfrak{sl}_n)$, then $\mu = (\mu_1, \mu_2, \dots, \mu_n)$, for some $\mu_1, \dots, \mu_n \in \mathbb{Z}$, satisfying $\mu_1 \geq \mu_2 \geq \dots \geq \mu_n$ and $\sum_{i=1}^n \mu_i = 0$. Assume $\mu \neq 0$, hence $(\mu_1, \dots, \mu_n) \neq (0, \dots, 0)$. If $\mu_1 < 0$, then $\mu_i < 0$, for all $2 \leq i \leq n$ and $\sum_{i=1}^n \mu_i \neq 0$, a contradiction. Thus we may assume $\mu_1 \geq 0$.

Now suppose there exists $\sigma \in S_n$ such that $\wp(\sigma(\tilde{\alpha} + \rho) - \rho - \mu) > 0$. Proposition 1.1 implies $(\varpi_i, \sigma(\tilde{\alpha} + \rho) - \rho - \mu) \in \mathbb{N}$, for all $1 \leq i \leq n-1$. In particular, $(\varpi_1, \sigma(\tilde{\alpha} + \rho) - \rho - \mu) = a_{\sigma^{-1}(1)} - n + 1 - \mu_1 \geq 0$ if and only if $\mu_1 \leq a_{\sigma^{-1}(1)} - n + 1$.

Observe that

$$a_{\sigma^{-1}(1)} - n + 1 = \begin{cases} 1 & \text{if } \sigma^{-1}(1) = 1 \\ -n & \text{if } \sigma^{-1}(1) = n \\ 1 - \sigma^{-1}(1) & \text{if } 1 < \sigma^{-1}(1) < n. \end{cases}$$

Then $(\varpi_1, \sigma(\tilde{\alpha} + \rho) - \rho - \mu) > 0$ if and only if $\sigma^{-1}(1) = 1$ and $\mu_1 \leq 1$. Since $\mu_1 \geq 0$, we have that $\mu_1 \in \{0, 1\}$. If $\mu_1 = 0$, then $\sum_{i=1}^n \mu_i = 0$ if and only if $\mu = 0$, a contradiction. Therefore $\mu_1 = 1$.

Now observe that $(\varpi_{n-1}, \sigma(\tilde{\alpha} + \rho) - \rho - \mu) = \sum_{i=1}^{n-1} (a_{\sigma^{-1}(i)} - n + i - \mu_i) = \mu_n - a_{\sigma^{-1}(n)}$. Hence $(\varpi_{n-1}, \sigma(\tilde{\alpha} + \rho) - \rho - \mu) \geq 0$ if and only if $\mu_n \geq a_{\sigma^{-1}(n)}$. If $1 < \sigma^{-1}(n) < n$, then $a_{\sigma^{-1}(n)} = n - \sigma^{-1}(n) \geq 1$, and hence $\mu_n \geq 1$. Then $1 = \mu_1 \geq \mu_2 \geq \dots \geq \mu_n \geq 1$ implies that $\mu_i = 1$, for all $1 \leq i \leq n$ and so $\sum_{i=1}^n \mu_i \neq 0$, a contradiction. Therefore $\sigma^{-1}(n) = n$, and $\mu_n = -1$.

Observe that since $\mu_1 = 1$, $\mu_n = -1$, and $\mu_1 \geq \mu_2 \geq \dots \geq \mu_n$, we have that $\mu_i \in \{1, 0, -1\}$, for all $2 \leq i \leq n-1$. If $\mu_2 = -1$, then $\mu = (1, -1, \dots, -1)$ and $\sum_{i=1}^n \mu_i \neq 0$, a contradiction. Suppose $\mu_2 = 1$. Since $\sigma^{-1}(1) = 1$ and $\sigma^{-1}(n) = n$, we have that $1 < \sigma^{-1}(2) < n$. Hence $(\varpi_2, \sigma(\tilde{\alpha} + \rho) - \rho - \mu) = a_{\sigma^{-1}(2)} - n + 1 = 1 - \sigma^{-1}(2) < 0$, a contradiction. Thus $\mu_2 = 0$.

Notice that if $\mu_j = -1$, for some $2 < j \leq n-1$, then $\mu_i = -1$, for all $j < i \leq n-1$. In which case $\sum_{i=1}^n \mu_i \neq 0$, giving rise to a contradiction. So $\mu_i = 0$, for all $2 \leq i \leq n-1$, and thus $\mu = (1, 0, \dots, 0, -1) = \tilde{\alpha}$.

(\Leftarrow) Follows from Proposition 4.1. \square

Proof of Theorem 4.1. Follows from Proposition 4.1 and the contrapositive of Proposition 4.2. \square

The following corollary is fundamental in Lie theory. We give an alternate proof of this well known result by using Weyl alternation sets.

Corollary 4.1. *If $\mu \in P(\mathfrak{sl}_n)$, then $m(\tilde{\alpha}, \mu) = \begin{cases} r & \text{if } \mu = 0 \\ 1 & \text{if } \mu \in \Phi \\ 0 & \text{otherwise.} \end{cases}$*

Proof. By Proposition 3.1.20 in [1], if $\mu \in P(\mathfrak{sl}_n)$, then there exists $w \in W$ and $\xi \in P_+(\mathfrak{sl}_n)$ such that $w(\xi) = \mu$. Also by Proposition 3.2.27 in [1] we know that weight multiplicities are invariant under W . Thus it suffices to compute $m(\tilde{\alpha}, \mu)$ for $\mu \in P_+(\mathfrak{sl}_n)$. By Corollary 3.1, $m(\tilde{\alpha}, 0) = r$. By Theorem 4.1 we know $\mathcal{A}(\tilde{\alpha}, \tilde{\alpha}) = \{1\}$, and $\mathcal{A}(\tilde{\alpha}, \mu) = \emptyset$ whenever $\mu \in P_+(\mathfrak{sl}_{n+1}) - \{0, \tilde{\alpha}\}$. This implies that $m(\tilde{\alpha}, \tilde{\alpha}) = \wp(1(\tilde{\alpha} + \rho) - \rho - \tilde{\alpha}) = \wp(0) = 1$, and that $m(\tilde{\alpha}, \mu) = 0$ whenever $\mu \in P_+(\mathfrak{sl}_{n+1}) - \{0, \tilde{\alpha}\}$. \square

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